On Certain Properties of the Plane Cubic Curve in Relation to the Circular Points at Infinity.

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Part II.—*On Certain Plane Cubic Curves and their Angles of Intersection, with some Account of Conics Cutting Orthogonally.*

I propose to investigate here some methods of generating certain plane cubic curves in such a way that their angles of intersection assume a simple form.

Let us consider a system of conics passing through four fixed points. Let U, V be any two conics of the system, where

$$U = ax^2 + by^2 + 2hxy + 2gx + 2fy + c,$$
$$V = a'x^2 + b'y^2 + 2h'xy + 2g'x + 2f'y + c',$$

the axes being rectangular, then

$$U - \lambda V = 0 \tag{1}$$

is any conic of the system. Now, let us seek the locus of the points of contact of tangents of (1) drawn parallel to the axis of x. For these two points we must have

$$\frac{dU}{dx} - \lambda \frac{dV}{dx} = 0, \text{ or } L - \lambda L' = 0, \tag{2}$$

if L, M, N are the three first differential coefficients of U and L', M', N' the similar functions for V. Hence, for the locus we have

$$2\phi\, U = \frac{dV}{dx} - V\frac{dU}{dx} = 0. \tag{3}$$

Similarly, for the locus of the points of contact of the tangents parallel to the axis of y, we have

$$2\psi = U\frac{dV}{dy} - V\frac{dU}{dy} = 0, \tag{4}$$

and for the locus of the points of contact of tangents parallel to the line $lx + my = 0$, we have

$$l\psi - m\phi = 0. \tag{5}$$

These curves are evidently cubics passing through the four points $U = 0$, $V = 0$. Further, they pass through the intersection of the diagonals and opposite sides of the quadrangle formed by the four points; for, putting

$$2U = Lx + My + N,$$
$$2V = L'x + M'y + N',$$

as is identically the case, it is readily seen that both (3) and (4) are satisfied by

$$\frac{L'}{L} = \frac{M'}{M} = \frac{N'}{N}, \tag{6}$$

which gives the vertices of the common self-conjugate triangle of U, V, namely, the three points just mentioned.

Differentiating (3) and (4), we have

$$\frac{d\phi}{dx} = a'U - aV,\quad \frac{d\phi}{dy} = h'U - hV + 2J, \tag{7}$$

$$\frac{d\psi}{dx} = h'U - hV - 2J,\quad \frac{d\psi}{dy} = b'U - bV, \tag{8}$$

where

$$\frac{dV}{dx}\,\frac{dU}{dy} - \frac{dV}{dy}\,\frac{dU}{dx} = 4J. \tag{9}$$

Thus for the four points $U = V = 0$, we have $\frac{d\phi}{dx} = 0$; that is, the tangents to the cubic at these points are parallel to the axis of x. Again, these tangents are parallel to an asymptote of the cubic, for, seeking the points at infinity on (3), we obtain

$$\left.\begin{aligned}&(a'x + h'y)(ax^2 + 2hxy + by^2) - (ax + hy)(a'x^2 + b'y^2 + 2h'xy) = 0,\\ \text{or}\quad &y\,\{(a'x + h'y)(hx + by) - (ax + hy)(h'x + b'y)\} = 0.\end{aligned}\right\} \tag{10}$$

The curve ϕ is then a general cubic described in a unique manner corresponding to a given direction; that is, the cubic passes through the four points so that its tangents at these points are parallel to an asymptote, there being but one cubic corresponding to a given direction, as we see by considering the fact that the cubic passes through the intersections of the diagonals and opposite sides. The latter points are always on the cubic, when the tangents at the four points pass through a point P on the curve (see Salmon's "Higher Plane Curves," §150), the tangents at the three points passing through the tangential of P. Hence, the cubics ϕ, ψ cut at right angles at the four points $U = V = 0$, and the cubics $l\psi - m\phi = 0$, $l'\psi - m'\phi = 0$ cut, at the same four points, at an angle equal to that between the lines $lx + my = 0$, $l'x + m'y = 0$. It may be observed that the cubics ϕ, ψ or the cubics $l\psi - m\phi$, $l'\psi - m'\phi$ intersect at two other points beside the seven points already determined. These two points are the directions of the two other asymptotes, which are common to all the cubics, as we see from (10). These two directions are evidently those of the axes of the two parabolæ which can be described through the four points. Let us now seek the angles of intersection of the cubics ϕ, ψ at the vertices of the common self-conjugate triangle of U, V. For these points we have $J = 0$ in (7) and (8), and if $U - \lambda V = 0$, λ, is one of the roots of the cubic obtained by taking the discriminant of $U - \lambda V$, namely,

$$\Delta - \Theta\lambda + \Theta'\lambda^2 - \Delta'\lambda^3 = 0. \tag{11}$$

We thus obtain

$$\tan\theta = \frac{\dfrac{d\phi}{dx}\dfrac{d\psi}{dy} - \dfrac{d\phi}{dy}\dfrac{d\psi}{dx}}{\dfrac{d\phi}{dx}\dfrac{d\psi}{dx} + \dfrac{d\phi}{dy}\dfrac{d\psi}{dy}} = \frac{(\lambda a' - a)(\lambda b' - b) - (\lambda h' - h)^2}{(\lambda h' - h)\{(a' + b')\lambda - (a + b)\}}. \tag{12}$$

Suppose $h = h' = 0$, then the two cubics corresponding to the two rectangular directions parallel to the axes of coordinates cut at right angles at the vertices of the common self-conjugate triangle of the conics. In this case one of the conics, V, say, is a circle, and the axes of all the conics of the system are parallel to the axes of coordinates. Thus the two cubics in this case are the loci of the vertices of a system of conics passing through four points on a circle. I $V = x^2 + y^2 - k^2$, these loci are

$$(a - b)xy^2 + g(y^2 - x^2) - 2fxy - k^2(ax + g) - cx = 0, \tag{13}$$

$$(a - b)x^2y + f(y^2 - x^2) + 2gxy + k^2(by + f) + cy = 0, \tag{14}$$

so that they touch infinity and meet it again in a rectangular direction. We have thus proved that the cubics (13) and (14) cut at right angles at their seven finite points of intersection.

Now, suppose $a + b = a' + b' = 0$ in (12), then, as in the preceding case the cubics ϕ, ψ cut at right angles at the three points. All the conics passing through the four points are then equilateral hyperbolæ, so that one of the points is the intersection of the perpendiculars of the triangle formed by the other three points. Considering then a triangle formed by three points of intersection of U and V, we see that the cubics intersect at the intersection of the perpendiculars and the feet of the perpendiculars, besides the three points themselves. Again, for the two points at infinity common to the cubics, we have, from (10),

$$(ax + hy)(h'x - a'y) - (hx - ay)(a'x + h'y) = 0$$

or
$$(ah' - a'h)(x^2 + y^2) = 0\,; \tag{15}$$

that is, the two cubics are circular. But two circular cubics cutting each other at right angles at their seven finite points of intersection are confocal. Hence, we see that the two curves ϕ, ψ are in this case confocal circular cubics. Suppose we consider now, in the same case, two cubics corresponding to directions which are not at right angles to each other. For the angles of intersection of $l\psi - m\phi$, $l'\psi - m'\phi$ at the feet of the perpendiculars of the triangle formed by three points of intersection of the equilateral hyperbolæ, U, V, we have, instead of (12),

$$\text{an}\ \theta = \frac{(lm' - l'm)\left(\frac{d\phi}{dx}\,\frac{d\psi}{dy} - \frac{d\phi}{dy}\,\frac{d\psi}{dx}\right)}{ll'\left\{\left(\frac{d\psi}{dx}\right)^2 + \left(\frac{d\psi}{dy}\right)^2\right\} + mm'\left\{\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\phi}{dy}\right)^2\right\} - (lm' + l'm)\left\{\frac{d\phi}{dx}\,d\psi + \frac{d\phi}{dy}\,\frac{d\psi}{dy}\right.}$$

But from (7) and (8) we have

$$\frac{d\phi}{dx} + \frac{d\psi}{dy} = 0, \quad \frac{d\phi}{dy} - \frac{d\psi}{dx} = 0\,,$$

which gives
$$\tan\theta = \frac{lm' - l'm}{ll' + mm'}\,; \tag{16}$$

that is, the angle θ is equal to the angle between the real asymptotes of the circular cubics. The two circular cubics thus cut at the same angle at all their seven finite points of intersection. It may be observed that if S_1, S_2, S_3 are the circles

described on the sides of the triangle formed by three points of intersection of the equilateral hyperbolæ as diameters, one of the cubics can be written

$$lS_2S_3 + mS_3S_1 + nS_1S_2 = 0, \tag{17}$$

where $l + m + n = 0$, for this represents a circular cubic passing through the vertices, the intersection of the perpendiculars and the feet of the perpendiculars.

Let us now consider the locus of the points of contact of tangents drawn from a fixed point to the system of conics passing through four fixed points. If Q, P are the polars of the fixed point with regard to the conics U, V respectively, the locus is found to be the cubic

$$PU - QV = 0. \tag{18}$$

This cubic, like the preceding ones, passes through the four points $U = 0$, $V = 0$, and the intersections of diagonals and opposite sides of the quadrangle formed by the four points, namely—the three latter points—the vertices of the common self-conjugate triangle of the conics U, V. Let us now consider the angles of intersection of cubics corresponding to two different points at the seven points of intersection that we have just mentioned. Let the axes be taken so that the two points are $\pm k,\ 0$, then $P = N' + kL'$, $P' = N' - kL'$, $Q = N + kL$, $Q' = N - kL$.

For one of the four points given by $U = V = 0$, we get

$$\cot\theta = \frac{(N'^2 - k^2L'^2)(L^2 + M^2) + (N^2 - k^2L^2)(L'^2 + M'^2) - 2(NN' - k^2LL')(LL' + MM')}{2k\,LN' - L'N)(LM' - L'M)}$$

$$= \frac{(LN' - NL')^2 + (MN' - M'N)^2 - k^2(LM' - L'M)^2}{2k(LN' - L'N)(LM' - L'M)};$$

but we have $\quad Lx + My + N = 0, \quad L'x + M'y + N' = 0,$

from $U = 0$, $V = 0$, and from these we obtain

$$\frac{x}{MN' - MN'} = \frac{y}{NL' - N'L} = \frac{1}{LM' - L'M},$$

so that we get

$$\cot\theta = \frac{x^2 + y^2 - k^2}{2ky}; \tag{19}$$

that is, the angle of intersection at 0, one of the four points $U = V = 0$, is equal to the angle subtended at 0 by the points $\pm k$, 0, A, B, say. This is also apparent from the fact that the tangents to the cubic (18) at the four points all pass through the point A, which is on the curve, a result that can be proved by projecting what we have already arrived at with regard to the locus of the points of contact of parallel tangents. Hence two cubics such as (18) corresponding to two points A, B will cut orthogonally at such of the four points as are made to lie on the circle described on AB as diameter; for instance, if we have a system of conics passing through four points on a circle, two cubics corresponding to the points A, B at the extremities of a diameter of the circle will cut orthogonally at the four points. I now seek the angles of intersection of the cubics at the vertices of the common self-conjugate triangle of the conics U, V.

Putting $V = \lambda U$, where λ is determined by the cubic (11), we easily find

$$\cot\theta = \frac{(\lambda g - g')^2 + (\lambda f - f')^2 - k^2(\lambda a - a')^2 - k^2(\lambda h - h')^2}{2k\{(\lambda a - a')(\lambda f - f') - (\lambda g - g')(\lambda h - h')\}}. \tag{20}$$

This expression simplifies if the two points lie on the locus of the centres of the conics of the system, namely, the conic passing through the middle points of all the lines joining the four points. We may then take U so as to have its centre at k, 0 and V its centre at $-k$, 0. This gives $ka + g = 0$, $kh + f = 0$, $-ka' + g' = 0$, $-kh' + f' = 0$, whence

$$g^2 + f^2 = k^2(a^2 + h^2), \quad af - gh = 0, \tag{21}$$

$$g'^2 + f'^2 = k^2(a'^2 + h'^2), \quad a'f' - g'h' = 0, \tag{22}$$

so that (20) becomes

$$\cot\theta = \frac{aa' + hh'}{a'h - ha'}, \tag{23}$$

or

$$\frac{gg' + ff'}{f'g - fg'}, \tag{24}$$

that is, the angles of intersection at the three points are the same and equal to the angle between the polars of the origin with regard to the two particular conics U, V, of the system. In this case the circle circumscribing the common self-conjugate triangle passes through the tangentials A', B' of A, B with regard to the two cubics, respectively, for the tangents at the vertices to the cubics are concurrent at A', B', respectively. Hence, if A', B' are at the extremities of a diameter of the circle, the cubics cut orthogonally at the vertices of the triangle.

I have noticed that in the case of a system of conics passing through four concyclic points the locus of the vertices consists of two cubics. The locus of the foci in the same case, as has been shown by Sylvester, consists of the two circular cubics having the four points for foci. It is thus suggested that the locus of points on the major axis of the conics, at a distance x from the centre given by $x^2 = a^2 + kb^2$, where a, b are the semiaxes and k is a constant, is also a cubic, and this, in fact, is found to be the case. Let S, S' denote the two cubics (13) and (14), then the equations of two such loci corresponding to the axes of x and y, respectively, are found to be

$$S' + kY = 0, \; S + k'X = 0, \tag{25}$$

where Y and X are, respectively, the equations of the three parallels to the axes of x and y drawn through the vertices of the common self-conjugate triangle of the conics. The confocal circular cubics, of course, correspond to $k = k' = -1$, and by taking $k = k' = 1$, we get the locus of the points where the director circles are met by the axes of the conics.

I now consider the problem of describing a cubic through six points on a conic, so that the tangents at these points all pass through a point P, that is, so that the conic is the polar conic of the point P with regard to the cubic. If we consider two solutions of this problem and P, Q are the two corresponding points the angle of intersection at one of the six points O is evidently the angle subtended at O by PQ. Also if P, Q be taken on the line at infinity, the angles of intersection at all the six points are equal to the angle between the directions P, Q. Let x, y be the tangents from P to the conic, and z the chord of contact, then the conic may be taken as $z^2 - xy = 0$, and the cubic is easily seen to be of the form

$$U = z^3 - 3xyz + ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0. \tag{26}$$

For the six points of intersection of the cubic and the conic, putting $x = 1$, $y = \theta^2$, $z = \theta$, we get

$$d\theta^6 + 3c\theta^4 - 2\theta^3 + 3b\theta^2 + a = 0, \tag{27}$$

in which it is to be observed that the coefficients of θ^5 and θ are absent. Now ∞, 0 are the values of θ corresponding to the tangents x, y drawn from P to the conic. Thus we see that these values are such that the corresponding factors being taken as lines of reference the binary sextic giving the six points on the

conic wants its second and penultimate terms. Now if we have a binary sextic $U=(a_0, a_1, a_2, a_3, a_4, a_5, a_6 \between x, y)^6$ and transform to a new set of variables and make the coefficients of the second and penultimate terms vanish we get

$$x' \frac{du}{dx} + y' \frac{du}{dy} = 0, \quad x \frac{du'}{dx'} + y \frac{du'}{dy'} = 0, \tag{28}$$

from which by elimination of x', y' we get an equation in x, y of the 26^{th} degree; but this must be divisible by U as (28) are satisfied by $xy' - yx' = 0$, $u = u' = 0$. Thus the result of elimination is of the twentieth degree, that is, it consists of ten quadratic factors. We thus see that there are ten cubics and ten points such as P. Let us now consider two of the cubics, taking the lines joining P, Q and the pole of PQ with regard to the conic as triangle of reference. Let the conic be

$$U = ax^2 + by^2 + cz^2 + 2hxy, \tag{29}$$

then if U_1, U_2 are the two cubics, $\frac{dU_1}{dx}$ and $\frac{dU_2}{dy}$ must be proportional to U. Hence integrating we get

$$\begin{aligned} U_1 &= 3xU - (2ax^3 + 3hx^2y) + \alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3, \\ U_2 &= 3yU - (2by^3 + 3hxy^2) + \alpha' x^3 + \beta' x^2 z + \gamma' x z^2 + \delta' z^3; \end{aligned} \tag{30}$$

but U_1 and U_2 intersect on U, so that we must have

$$\begin{aligned} l\,(\alpha y^3 + \beta y^2 z + \gamma y z^2 + \delta z^3) &- l\,(2ax^3 + 3hx^2 y) \\ &+ m\,(\alpha' x^3 + \beta' x^2 z + \gamma' x z^2 + \delta' z^3) - m\,(2by^3 + 3hxy^2) \\ &= (\lambda x + \mu y + \nu z)(cz^2 + ax^2 + by^2 + 2hxy) \end{aligned} \tag{31}$$

from which, by comparison of the coefficients of z, we obtain

$$l\beta y^2 + m\beta' x^2 = \nu\,(ax^2 + by^2 + 2hxy),$$

which gives $\beta = \beta' = \gamma = 0$. Hence as the cubics intersect again at three points on the line $\lambda x + \mu y + \nu z$, which takes the form $\lambda x + \mu y = 0$, we see that the cubics U_1, U_2 intersect again at three points lying on the line passing through the pole of PQ with regard to the conic. Let z be at infinity and x, y rectangular coordinates then U_1, U_2 cut orthogonally at six points on U.

In the case in which the tangents at four points on the cubic pass through a point not on the curve, then if $x = l\alpha$, $y = m\beta$, $z = n\gamma$, where α, β, γ are the

perpendiculars from a point on the sides of the triangle of reference, we may take $x^2 = y^2 = z^2$ as the four points, and then if a, b, c is the intersection of the tangents thereat, the cubic takes the form

$$a^2 (by + cz)(y^2 - z^2) + b^2 (cz + ax)(z^2 - x^2) + c^2 (ax + by)(x^2 - y^2) \\ + \theta \{ax (y^2 - z^2) + by (z^2 - x^2) + cz (x^2 - y^2)\} = 0, \tag{32}$$

where θ is arbitrary. The polar conic of a, b, c is, it may be observed,

$$(3a^2 + \theta)(b^2 - c^2)x^2 + (3b^2 + \theta)(c^2 - a^2)y^2 + (3c^2 + \theta)(a^2 - b^2)z^2 = 0, \tag{33}$$

which, of course passes through the four points and also meets the cubic again at two points lying on the line

$$3 (a^2 - b^2)(b^2 - c^2)(c^2 - a^2)(ax + by + cz) + ax (b^2 - c^2)(\theta + 3a^2)(\theta - a^2) \\ + by (c^2 - a^2)(\theta + 3b^2)(\theta - b^2) + cz (a^2 - b^2)(\theta + 3c^2)(\theta - c^2) = 0, \tag{34}$$

which, when θ varies, envelopes a conic.

Hence, if we take two cubics, such as (32), corresponding to two points, P, P', the angle of intersection at one of the four points, O say, is equal to the angle subtended at O by PP'.

I observe here that if a cubic be referred to an inscribed triangle, the tangents at whose vertices pass through a point a, b, c, the equation of the curve can be written

$$bcx (y^2 - z^2) + cay (z^2 - x^2) + abz (x^2 - y^2) + 2mxyz = 0, \tag{35}$$

in which case we see that the lines joining the vertices to the points in which the opposite sides respectively meet the curve again, pass through a point, namely, bc, ca, ab. If $m = 0$ in (35), we have a cubic already considered, namely, that on which the four points $x^2 = y^2 = z^2$ are such that the tangents thereat intersect in the point bc, ca, ab on the curve, and the tangents at the vertices of the triangle of reference, namely, the intersections of diagonals and opposite sides of the quadrangle formed by the four points, intersect at the point a, b, c, the tangential of bc, ca, ab. If these cubics are both circular, it may be observed that the *satellites* ε, ζ are either parallel or at right angles to each other.

Again, let α, β, γ, δ denote the lengths of the perpendiculars from a point on four lines, then it is readily seen that two cubics cutting each other at right angles at the three collinear points $\alpha\delta$, $\beta\delta$, $\gamma\delta$ can be written

$$\left.\begin{aligned} (\alpha + \delta)(\beta + \delta)(\gamma + \delta) - k\delta^2\varepsilon = 0, \\ (\alpha - \delta)(\beta - \delta)(\gamma - \delta) - k'\delta^2\zeta = 0. \end{aligned}\right\} \tag{36}$$

We can hence show that the cubics

$$(a+\lambda a')\,x\,(my^2-nz^2)+(b+\lambda b')\,y\,(nz^2-lx^2)$$
$$+\,(c+\lambda c')\,z\,(lx^2-my^2)+2pxyz=0, \quad (37)$$

where l, m, n, p are variable, are such that the tangents at the vertices of the triangle pass through a point. This point is given by $(a+\lambda a')\,x=(b+\lambda b')\,y=(c+\lambda c')\,z$, so that if λ is variable, it lies on the conic

$$(bc'-b'c)\,yz+(ca'-c'a)\,zx+(ab'-a'b)\,xy=0. \quad (38)$$

If this conic is made to coincide with the circumscribing circle, namely, if the points bc, ca, ab; $b'c'$, $c'a'$, $a'b'$ lie on the circumscribing circle, two cubics intersect at the vertices at an angle equal to that subtended by the two points at any point of the circle.

I now mention a case in which two cubics intersect orthogonally at six points of their intersection. Let u, v be circular coordinates; that is, let $x+iy=u$, $x-iy=v$, where x, y are rectangular coordinates, then let us consider the quartic curve

$$\phi=a\,(u^4+v^4)+b\,(u^3+v^3)+c\,(u^2+v^2)+lu^2v^2+muv$$
$$+\,\alpha uv\,(u^2+v^2)+\beta uv\,(u+v). \quad (39)$$

Now, let us take ϕ, so that $\dfrac{d\phi}{du}-\dfrac{d\phi}{dv}$ is proportional to $(u-v)\dfrac{d^2\phi}{du\,dv}$. We thus get

$$4a\,(u^2+v^2+uv)+3b\,(u+v)+2c=k\{4luv+m+3\alpha\,(u^2+v^2)+2\beta\,(u+v)\}, \quad (40)$$

whence $\alpha=\dfrac{4a}{3k+1}$, $\beta=\dfrac{3b}{2k+1}$,

$$l=\frac{6a\,(k+1)}{(2k+1)(3k+1)},\quad m=\frac{2c}{k+1}. \quad (41)$$

It then readily follows that the cubics $\dfrac{ld\phi}{dx}+\dfrac{md\phi}{dy}$, $\dfrac{l'd\phi}{dx}+\dfrac{m'd\phi}{dy}$, where $ll'+mm'=0$ or, which is the same thing, the cubics $\dfrac{d\phi}{du}\pm\dfrac{\lambda d\phi}{dv}$, cut orthogonally at six points on the conic $\dfrac{d^2\phi}{dudv}$ or $\dfrac{d^2\phi}{dx^2}+\dfrac{d^2\phi}{dy^2}$, namely,

$$4a\,(u^2+v^2+uv)+3b\,(u+v)+2c=0,$$

or
$$4a\,(3x^2 - y^2) + 3bx + c = 0. \tag{42}$$

Let us now consider a cubic and a conic cutting each other orthogonally at their six points of intersection. Since a cubic and a conic together involve 14 constants and 6 conditions are involved in the orthogonality, there only remain 8 constants, which thus are one too few in number to give a general cubic involving nine constants. It hence appears that a cubic must satisfy one relation with the circular points at infinity if it is capable of being cut orthogonally by a conic at six points. I now propose to determine such a cubic when the conic is given.

Let the conic referred to its principal axes be $\alpha x^2 + \beta y^2 + 1$, then if the cubic is ϕ, we must evidently have

$$\alpha x \frac{d\phi}{dx} + \beta y \frac{d\phi}{dy} + k\phi = (lx + my + n)(\alpha x^2 + \beta y^2 + 1). \tag{43}$$

Now let $\phi = \phi_3 + \phi_2 + \phi_1 + \phi_0$, where ϕ_n is a rational integral homogeneous expression in x, y of the n^{th} degree, then we must have

$$\left.\begin{aligned}
&\alpha x \frac{d\phi_3}{dx} + \beta y \frac{d\phi_3}{dy} + k\phi_3 = (lx + my)(\alpha x^2 + \beta y^2)\,,\\
&\alpha x \frac{d\phi_2}{dx} + \beta y \frac{d\phi_2}{dy} + k\phi_2 = n\,(\alpha x^2 + \beta y^2),\\
&\alpha x \frac{d\phi_1}{dx} + \beta y \frac{d\phi_1}{dy} + k\phi_1 = lx + my,\\
&\qquad\qquad\qquad\qquad\quad k\phi_0 = n.
\end{aligned}\right\} \tag{44}$$

In the case of ϕ_2 there is an ambiguity, for if $\phi_2 = ax^2 + by^2 + 2hxy$, we have from the coefficient of xy, $h\,(\alpha + \beta + k) = 0$, which gives either $h = 0$ or $\alpha + \beta + k = 0$. In the first case the cubic is

$$\frac{l\alpha x^3}{3\alpha + k} + \frac{m\alpha x^2 y}{2\alpha + \beta + k} + \frac{l\beta x y^2}{2\beta + \alpha + k} + \frac{m\beta y^3}{3\beta + k} + \frac{n\alpha x^2}{2\alpha + k} + \frac{n\beta y^2}{2\beta + k} + \frac{lx}{\alpha + k} + \frac{my}{\beta + k} + \frac{n}{k} = 0, \tag{45}$$

and in the second case

$$\frac{l\alpha x^3}{2\alpha - \beta} + mx^2y + lxy^2 + \frac{m\beta y^3}{2\beta - \alpha} + n\,\frac{(\alpha x^2 - \beta yz)}{\alpha - \beta} + 2hxy - \frac{lx}{\beta} - \frac{my}{\alpha} - \frac{n}{\alpha + \beta} = 0. \tag{46}$$

The latter cubic, it may be observed, can be written

$$lx\left\{\frac{ax^2}{2\alpha-\beta}+y^2-\frac{1}{\beta}\right\}+my\left\{x^2+\frac{\beta y^2}{2\beta-\alpha}-\frac{1}{\alpha}\right\}$$
$$+n\left\{\frac{ax^2-by^2}{\alpha-\beta}-\frac{1}{\alpha+\beta}\right\}+2hxy=0\,, \tag{47}$$

in which the three conics multiplied by lx, my and n, respectively, are all confocal with the given conic $\alpha x^2+\beta y^2+1=0$. Again, if the cubic (45) be written

$$lxU_1+myU_2+nU_3=0\,, \tag{48}$$

the three conics U_1, U_2, U_3 are confocal with $\alpha x^2+\beta y^2+1$, corresponding to three values, $\frac{1}{2}(k+\alpha)$, $\frac{1}{2}(k+\beta)$, $\frac{1}{2}k$, respectively, of λ in the equation

$$\frac{\alpha x^2}{\lambda+\alpha}+\frac{\beta y^2}{\lambda+\beta}+\frac{1}{\lambda}=0\,. \tag{49}$$

Similarly in the case of the sphere we can show that the most general equation of a cubic cone cutting orthogonally the cone $ax^2+by^2+cz^2=0$ is

$$lxU_1+myU_2+nzU_3=0\,, \tag{50}$$

where U_1, U_2, U_3 are the three cones confocal with the given one, which are obtained by taking λ equal to $\frac{1}{2}(3k-a)$, $\frac{1}{2}(3k-b)$, $\frac{1}{2}(3k-c)$ respectively, in

$$\frac{ax^2}{a-\lambda}+\frac{by^2}{b-\lambda}+\frac{cz^2}{c-\lambda}=0. \tag{51}$$

Again, if $3k=a+b+c$, the most general equation of the orthogonal cubic cone is

$$lxU_1+myU_2+nzU_3+pxyz=0\,. \tag{52}$$

As an extension of the preceding results I may notice that we can obtain in the same way the equation of a cubic surface cutting a given quadric orthogonally along the entire curve of intersection. If the cubic be written $\phi_3+\phi_2+\phi_1+\phi_0$, where ϕ_n is a homogeneous expression in x, y, z of the n^{th} degree, we have if the quadric is $ax^2+by^2+cz^2+d$,

$$\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}\right)(\phi_3+\phi_2+\phi_1+\phi_0)+k(\phi_3+\phi_2+\phi_1+\phi_0)$$
$$=(lx+my+nz+p)(ax^2+by^2+cz^2+d)\,. \tag{53}$$

We thus obtain the equation of the cubic in the form

$$lxU_1 + myU_2 + nzU_3 + pU_4 = 0, \tag{54}$$

where U_1, U_2, U_3, U_4 are four confocal quadrics involving the parameter k. If k have a certain determinate value we may add to (54) the term $qxyz$, and further, if there is a relation connecting a, b, c, a term such as rxy or ryz or rzx may be added.

I now proceed to consider some properties of confocal nodal circular cubics. Such a curve involves six constants, so that when the two foci are given, there are still two indeterminate parameters. By inverting the equation of a conic from a point on itself, we get the equation of such a cubic in the form

$$PB\rho_1 \pm PA\rho_2 + (PA \pm PB)\rho = 0, \tag{55}$$

where A, B are the foci, P is the node, and ρ_1, ρ_2, ρ are the distances of a point from A, B, P, respectively.

It may be observed that A, B, may be imaginary, that is, A, B may be the antipoints of two real points, A', B', in which case, the points A', B' are foci, and the equation (55) gives another solution of the problem to describe a nodal circular cubic with given foci and node. Thus there are four solutions of the problem altogether. Now let us make use of elliptic coordinates, the points A, B being taken as the foci of the system of confocal conics, that is, let the conics be

$$\frac{x^2}{\mu^2} + \frac{y^2}{\mu^2 - c_2} = 1, \frac{x^2}{\nu^2} - \frac{y^2}{c^2 - \nu^2} = 1,$$

whence

$$cx = \mu\nu,\ cy = \surd\,\{(\mu^2 - c^2)(c^2 - \nu^2)\}. \tag{56}$$

Now let the node of (55) lie on the axis of y, then from (55) we have

$$\rho_1 + \rho_2 = 2\rho,\ \text{or}\ \mu = \rho, \tag{57}$$

whence $\mu^2 = x^2 + (y - \beta)^2$, where 0, β are the coordinates of the node. From this we obtain

$$\surd\,(c^2 - \nu^2)\{\mu \pm \surd\,(\mu^2 - c^2)\} = c\beta, \tag{58}$$

whence

$$\frac{d\mu}{\surd\,(\mu^2 - c^2)} \mp \frac{\nu d\nu}{c^2 - \nu^2} = 0. \tag{59}$$

Again consider the cubic with a node on the axis of x, the points A, B in this case in (55) being the imaginary foci of the system of conics. We have then

$$\mu^2 - c^2 = \rho'^2 = (x-\alpha)^2 + y^2, \tag{60}$$

whence

$$\nu\,\{\mu \pm \surd\,(\mu^2 - c^2)\} = c\alpha, \tag{61}$$

and

$$\frac{d\mu}{\surd\,(\mu^2-c^2)} \pm \frac{d\nu}{\nu} = 0. \tag{62}$$

Now if two curves are represented by the differential equations

$$Pd\mu + Qd\nu = 0,\ P'd\mu + Q'd\nu = 0,$$

and there is

$$PP'\,(\mu^2 - c^2) + QQ'\,(c^2 - \nu^2) = 0, \tag{63}$$

then these two systems of curves cut orthogonally. But this condition is satisfied for (59) and (62), so that the systems of cubics we have been considering cut orthogonally. This orthogonality occurs at only one point of intersection of the cubics, which, we see from (57) and (60), intersect on the line $\rho^2 - \rho'^2 = c^2$. It may be observed that the remaining four points of intersection of the two cubics lie on a point circle, for from (57) we have $\mu = \rho$, and for (60), we may substitute $\nu^2 - c^2 = \rho'^2$, whence $\mu^2 + \nu^2 - c^2 = \rho^2 + \rho'^2$, that is, $(x-\alpha)^2 + (y-\beta)^2 = 0$.

In the general case when the node of the cubic is at an arbitrary point, its equation in elliptic coordinates can be written

$$\{r\cos\alpha + \sin\alpha\,\nu\,(c^2 - \nu^2)\}\,\{\mu \pm \nu\,(\mu^2 - c^2)\} = cp, \tag{64}$$

or, in x, y coordinates

$$\begin{aligned} x^2 + y^2 - 4p\,(x\cos\alpha + y\sin\alpha) + 2p^2 + c^2\cos 2\alpha \\ = \rho\rho' = \pm\surd\,\{(x^2+y^2)^2 - 2c^2(x^2 - y^2) + c^4\}. \end{aligned} \tag{65}$$

Now when two curves are represented by $Pd\mu + Qd\nu = 0$, $P'd\mu + Q'd\nu = 0$, the angle θ between them is given by

$$\tan\theta = \frac{(PQ' - P'Q)\,\nu\,\{(\mu^2 - c^2)(c^2 - \nu^2)\}}{PP'\,(\mu^2 - c^2) + QQ'\,(c^2 - \nu^2)}. \tag{66}$$

Hence for the cubic (64) and a similar cubic with β instead of α we find $\theta = \alpha - \beta$, that is, the angle at one point of intersection is equal to the angle between the asymptotes, for from (65) we see that $x\cos\alpha + y\sin\alpha = 0$ is parallel to the real asymptote of the curve. This point is one of three points of intersection lying on

a line, as we see from (65), from which also we see that the other four points of intersection lie on a circle.

With respect to the system of circular cubics

$$lx(x^2+y^2+c^2)+my(x^2+y^2-c^2)+n(x^2+y^2)=0, \tag{67}$$

where l, m, n are variable, I observe that they all have the origin, which they pass through, as a double focus. Hence two of the system intersect elsewhere in four points, two points being real and two imaginary; the two latter being the antipoints of the former two.

Since (67) can be written in the form

$$\lambda\left(u+\frac{c^2}{u}\right)+\mu\left(v+\frac{c^2}{v}\right)+\nu=0,$$

where $x+iy=u$, $x-iy=v$, we can easily deduce that two cubics of the system (67) cut each other at all their five finite points of intersection at an angle equal to that between their asymptotes

$$lx+my+n=0, \quad l'x+m'y+n'=0.$$

With regard to the system of equilateral hyperbolæ

$$xy-ab-m(bx-ay)=0, \tag{68}$$

passing through the fixed points $\pm a$, $\pm b$, I observe that they are cut at right angles, at all the six points of intersection, by the system of cubics

$$\frac{x^3}{a}+\frac{y^3}{b}-3(ax+by)=c'. \tag{69}$$

I now proceed to consider some cases of conics cutting each other orthogonally at certain points of their intersection. First of all, I shall obtain a locus passing through all the points of intersection of two conics at which they cut orthogonally. Let $(a, b, c, f, g, h \between x, y, 1)^2=0$ be the equation of a conic U in rectangular Cartesian coordinates, and let $(A, B, C, F, G, H \between \lambda, \mu, \nu)^2=0$ be the corresponding tangential equation, then it is easy to see that

$$S=C(x^2+y^2)-2Gx-2Fy+A+B=0 \tag{70}$$

gives the director circle, viz., the locus of intersection of rectangular tangents of U. Hence, the director circle of $U_1 + \lambda U_2$ is

$$S_{11} + \lambda S_{12} + S_{22} = 0, \tag{71}$$

where

$$\begin{aligned} S_{12} = (a_1 b_2 + b_1 a_2 - 2h_1 h_2)(x^2 + y^2) - 2x(f_1 h_2 + f_2 h_1 - b_1 g_2 - b_2 g_1) \\ - 2y(g_1 h_2 + g_2 h_1 - a_1 f_2 - a_2 f_1) + c_1(a_2 + b_2) \\ + c_2(a_1 + b_1) - 2g_1 g_2 - 2f_1 f_2. \end{aligned} \tag{72}$$

The circle S_{12} is in fact the director circle of the covariant conic Φ, namely, the envelope of lines divided harmonically by U_1, U_2. Now, suppose the conics U_1, U_2 to cut orthogonally at a point P, then taking the origin at this point, we have $c_1 = c_2 = 0$, $g_1 g_2 + f_1 f_2 = 0$, and when these relations are satisfied, we see from (72) that S_{12} also passes through P. It may be observed that the equation of the director circle in circular coordinates is

$$S = Cxy - Gy - Fx + H = 0, \tag{73}$$

and that of S_{12} is

$$C_{12}xy - G_{12}y - F_{12}x + H_{12} = 0. \tag{74}$$

Hence, if U_1, U_2 cut orthogonally at all their four points of intersection, we must have $lU_1 + mU_2 = S_{12}$, or the circle S_{12} must vanish identically. Taking U_2 referred to its principal axes, we have

$$\begin{aligned} (a_1 b_2 + b_1 a_2)(x^2 + y^2) + 2b_2 g_1 x + 2a_2 f_1 y + c_1(a_2 + b_2) + c_2(a_1 + b_1) \\ = l(a_1 x^2 + b_1 y^2 + 2h_1 xy + 2g_1 x + 2f_1 y + c_1) + m(a_2 x^2 + b_2 y^2 + c_2), \end{aligned}$$

whence, by comparison of the coefficients of xy, x and y, we obtain

$$lh_1 = 0, \quad lg_1 = b_2 g_1, \quad lf_1 = a_2 f_1. \tag{75}$$

Taking $f_1 = g_1 = h_1 = 0$, we come upon the case of confocal conics. Again, let $l = 0$, in which case S_{12} vanishes identically; that is, $f_1 = g_1 = 0$, and h_1 remains indeterminate. In this case, if U_2 is $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$, U_1 is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{(a^2 - b^2)}{a^2 + b^2} + 2hxy = 0, \tag{76}$$

in which it is to be observed that the conic corresponding to $h=0$ is a confocal given by $a'^2b^2+b'^2a^2=0$. The equation (76) evidently denotes a system of conics passing through the vertices of a rhombus, which is real if the conic U_2 is a hyperbola.

Again from (75), we can take $h_1=0$, $l=a_2$ and $g_1=0$. This gives the system of conics

$$\frac{x^2}{a^2-2b^2}-\frac{y^2}{b^2}-1+2fy=0 \tag{77}$$

orthogonal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$, the conic corresponding to $f=0$ being a confocal given by $b'^2+b^2=0$. In this case, it may be observed, the four points of intersection lie on a circle passing through the four foci. In the same way we get the orthogonal system

$$\frac{x^2}{a^2}+\frac{y^2}{2a^2-b^2}+1+2gx=0. \tag{78}$$

If we want to find the condition that two conics should cut orthogonally at one point of intersection, we have to express that U_1, U_2 and S_{12} have a point in common. This gives (see Salmon's Conics) a relation of the form $T^2=64M$. This will be found to involve the coefficients of each of the conics U_1, U_2 in the eighth degree. It may be observed that the Jacobian of U_1, U_2, S_{12} gives a locus which passes through all the points at which U_1, U_2 cut orthogonally. Referring U_1, U_2 to their common self-conjugate triangle in trilinear coordinates, we have

$$U_1=l_1\alpha^2+m_1\beta^2+n_1\gamma^2,$$
$$U_2=l_2\alpha^2+m_2\beta^2+n_2\gamma^2,$$

and if
$$A=m_1n_2-m_2n_1,\quad B=n_1l_2-n_2l_1,\quad C=l_1m_2-l_2m_1,$$
$$A'=m_1n_2+m_2n_1,\quad B'=n_1l_2+n_2l_1,\quad C'=l_1m_2+l_2m_1,$$

we obtain

$$\begin{aligned}J= A'\alpha\cos A\,(B\gamma^2+C\beta^2)+B'\beta\cos B\,(C\gamma^2+A\alpha^2)+C'\gamma\cos C\,(A\beta^2+B\alpha^2)\\ +\{A'(B+C)+B'(C+A)+C'(A+B)\}\,\alpha\beta\gamma=0,\end{aligned} \tag{79}$$

this is the Hessian of another cubic, viz.

$$\begin{aligned}V= R\,(BCB'C'\alpha^3\cos B\cos C+CAC'A'\beta^3\cos C\cos A+ABA'B'\gamma^3\cos A\cos B)\\ -ABC(\alpha B'C'\cos B\cos C+\beta C'A'\cos C\cos A+\gamma A'B'\cos A\cos B)^3=0,\end{aligned} \tag{80}$$

where

$$R = AB'^2C'^2 \cos^2 B \cos^2 C + BC'^2A'^2 \cos^2 C \cos^2 A + CA'^2B'^2 \cos^2 A \cos^2 B$$
$$- A'B'C' \cos A \cos B \cos C \{A'(B + C) + B'(C + A) + C'(A + B)\}, \quad (81)$$

which represents another locus passing through all the points at which U_1, U_2 cut orthogonally. The conics U_1, U_2, S_{12} are polar conics of this cubic V.

It is to be observed that if U_1, U_2 cut orthogonally at two points, then U_1, U_2, S_{12} have two points in common and their Jacobian consists of a conic and a line intersecting in the two points, as we see by considering the Jacobian of three circles. Hence, in this case, we have

$$\frac{A'^2}{A} \cos^2 A = \frac{B'^2}{B} \cos^2 B,$$
$$\{A'(B + C) + B'(C + A) + C'(A + B)\}^2 = 4ABC'^2 \cos^2 C'\}, \quad (82)$$

in which case J is divisible by

$$\frac{A'\alpha \cos A}{A} + \frac{B'\beta \cos B}{B} = 0, \quad (83)$$

the line on which U_1, U_2 cut orthogonally. Of course there are two other similar pairs of relations obtained by interchanging α, β, γ, etc. If U_1, U_2 cut orthogonally at three points, we have, in addition to (82),

$$\frac{B'^2}{B} \cos^2 B = \frac{C'^2}{C} \cos^2 C,$$

in which case

$$J = (\lambda\alpha + \mu\beta)(\mu\beta + \nu\gamma)(\nu\gamma + \lambda\alpha), \quad (84)$$

where $\lambda = \dfrac{A' \cos A}{A}$, $\mu = \dfrac{B' \cos B}{B}$, $\nu = \dfrac{C' \cos C}{C}$.

In this case it can readily be shown that the three points of intersection of U_1, U_2 lie on the circular cubics

$$\frac{\alpha}{\cos A}(\beta^2 + \gamma^2 + 2\beta\gamma \cos A) \pm \frac{\beta}{\cos B}(\gamma^2 + \alpha^2 + 2\gamma\alpha \cos B)$$
$$\pm \frac{\gamma}{\cos C}(\alpha^2 + \beta^2 + 2\alpha\beta \cos C) = 0, \quad (85)$$

which curves, it is easy to see, are the loci of the foci of conics inscribed in the triangle and touching the radical axis of the circumscribed and nine-points circle

and the lines joining the feet of the perpendiculars respectively. Each of the three points lies on one of three properly selected cubics of the form comprised in (85). For a locus passing through all three points, we have, from (84), putting $A = A'^2 \cos^2 A$, $B = B'^2 \cos^2 B$, $C = C'^2 \cos^2 C$,

$$J = \left(\frac{\alpha}{A' \cos A} + \frac{\beta}{B' \cos B} + \frac{\gamma}{C' \cos C}\right)(A' \cos A\beta\gamma + B' \cos B\gamma\alpha + C' \cos C\alpha\beta) - \alpha\beta\gamma. \quad (86)$$

Now, let L be the polar of the intersection of the perpendiculars of the self-conjugate triangle with regard to the covariant conic Φ, namely,

$$\Phi = B'C'\alpha^2 + C'A'\beta^2 + A'B'\gamma^2,$$

then

$$L = \frac{\alpha}{A' \cos A} + \frac{\beta}{B' \cos B} + \frac{\gamma}{C' \cos C} \quad (87)$$

and

$$J(L, U_1, U_2) = A' \cos A\beta\gamma + B' \cos B\gamma\alpha + C' \cos C\alpha\beta, \quad (88)$$

and we get $J(U_1, U_2, S_{12})$ in the form

$$LJ(L, U_1, U_2) - kJ(U_1, U_2, \Phi), \quad (89)$$

where k can be determined in terms of invariants of U_1, U_2 and the circular points. This cubic, of course, is another locus in addition to the circle

$$S_{12} = A'(\beta^2 + \gamma^2 + 2\beta\gamma \cos A) + B'(\gamma^2 + \alpha^2 + 2\gamma\alpha \cos B) + C'(\alpha^2 + \beta^2 + 2\alpha\beta \cos C). \quad (90)$$

Again, from geometrical considerations, we can obtain a circular cubic passing through all the points at which U_1, U_2 cut orthogonally. If, from any point P, tangents are drawn to the system of conics inscribed in a quadrilateral, they belong to a system in involution, of which the tangents to the two conics of the system which pass through P are the double lines. Hence, if the latter pair of lines are at right angles to each other, the tangents from P to one conic of the system must pass through the circular points at infinity, or, in other words, P must be a focus of that conic. Thus we see that the circular cubic, which is the locus of the foci of conics inscribed in the quadrilateral formed by the common tangents of U_1, U_2 passes through all the points at which U_1, U_2 cut orthogonally. Taking circular coordinates, the tangents drawn from the circular points

to the system of conics $\Sigma_1 + \lambda\Sigma_2$ touching the common tangents of ε_1, ε_2 are, respectively,

$$\left.\begin{aligned} C_1x^2 - 2G_1x + A_1 + \lambda\,(C_2x^2 - 2G_2x + A_2) = 0\,, \\ C_1y^2 - 2F_1y + B_1 + \lambda\,(C_2y^2 - 2F_2y + B_2) = 0\,, \end{aligned}\right\} \tag{91}$$

and the locus of the foci is

$$\begin{aligned}(C_1x^2 - 2Gx_1 + A_1)&(C_2y^2 - 2F_2y + B_2) \\ &- (C_2x^2 - 2G_2x + A_2)(C_1y^2 - 2F_1y + B_1) = 0\,,\end{aligned} \tag{92}$$

which represents a circular cubic, as the coefficient of x^2y^2 vanishes. If we take the origin at the focus of the parabola touching the four common tangents, the latter curve may be written $F_2\nu\,(\lambda + \mu) + H_2\lambda\mu = 0$, and (92) then becomes

$$Cxy\,(x - y) - 2\,(G - F)\,xy + Ay - Bx = 0. \tag{93}$$

Combining this with S_{12} (74), we get the conic

$$(F_{12}y + G_{12}x - H_{12})\{C_1\,(x - y) + 2\,(F_1 - G_1)\} + C_{12}\,(A_1y - B_1x) = 0. \tag{94}$$

If we now express that this conic and U_1, U_2, and S_{12} are connected by a linear relation we evidently obtain the systems of conditions which must be satisfied if U_1, U_2 cut orthogonally at three points of their intersection. When U_1, U_2 are referred to their common self-conjugate triangle, the circular cubic (92) is

$$\begin{aligned}\left(\frac{Al_1l_2\alpha}{\sin A} + \frac{Bm_1m_2\beta}{\sin B} + \frac{Cn_1n_2\gamma}{\sin C}\right)&(\beta\gamma\sin A + \gamma\alpha\sin B + \alpha\beta\sin C) \\ &= (\alpha\sin A + \beta\sin B + \gamma\sin C)(Al_1l_2\alpha^2\cot A \\ &\quad + Bm_1m_2\,\beta^2\cot B + Cn_1n_2\gamma^2\cot C).\end{aligned} \tag{95}$$

Eliminating $\alpha\beta\gamma$ between this and J (79) and introducing the relations $\dfrac{\alpha^2}{A} = \dfrac{\beta^2}{B} = \dfrac{\gamma^2}{C}$ we get the line

$$\begin{aligned}2\,(Al_1l_2 + Bm_1m_2 + Cn_1n_2)&(A'BC\cos A\alpha + B'CA\cos B\beta + C'AB\cos C\gamma) \\ &+ M\{\alpha\cos A\,(A^2l_1l_2 - B^2m_1m_2 - C^2n_1n_2) \\ &+ \beta\cos B\,(B^2m_1m_2 - C^2n_1n_2 - A^2l_1l_2) \\ &+ \gamma\cos C\,(C^2n_1n_2 - A^2l_1l_2 - B^2m_1m_2)\} = 0\,,\end{aligned} \tag{96}$$

where $$M = A'\,(B + C) + B'\,(C + A) + C'\,(A + B).$$

Hence, if U_1, U_2 cut orthogonally at two points, this line must be a chord of

intersection, and if they cut orthogonally at three points the three coefficients of the line must each vanish identically. Again, combining (95) with S_{12} (90) we obtain the conic, as at (94),

$$\left\{\frac{Al_1l_2\alpha}{\sin A}+\frac{Bm_1m_2\beta}{\sin B}+\frac{Cn_1n_2\gamma}{\sin C}\right\}\{(B'+C')\,\alpha\sin B\sin C$$
$$+(C'+A')\,\beta\sin C\sin A+(A'+B')\,\gamma\sin A\sin B\}$$
$$=(A'\sin^2 A+B'\sin^2 B+C'\sin^2 C)\{Al_1l_2\alpha^2\cot A$$
$$+Bm_1m_2\beta^2\cot B+Cn_1n_2\gamma^2\cot C\}. \tag{97}$$

If both the conics U_1, U_2 are parabolæ, the cubic (92) reduces to a circle, namely, as is well known, the circle circumscribing the triangle formed by their three finite common tangents, or, in other words, the nine-points circle of their common self-conjugate triangle. Hence, if the parabolæ cut orthogonally at two points, these points must be the intersections of the circle just mentioned and S_{12}. Again, if the parabolæ cut orthogonally at three points, these two circles must coincide. Now S_{12} (90) being of the form $A'\rho_1^2\sin^2 A+B'\rho_2^2\sin^2 B+C'\rho_3^2\sin^2 C=0$, where ρ_1, ρ_2, ρ_3 are the distances of a point from the vertices of the common self conjugate triangle, respectively, is readily seen to cut orthogonally the circumscribing circle. Hence, the self-conjugate triangle must be such that the nine-points circle cuts orthogonally the circumscribing circle. This gives

$$r^2=2R^2, \tag{98}$$

where r is the radius of the polar circle and R that of the circumscribing circle, or

$$\sin^2 A+\sin^2 B\sin^2 C=1. \tag{99}$$

The trilinear equations of the two parabolæ are then

$$\alpha^2\cot A\cot C+\beta^2\cot B\cot C+\gamma^2\pm\left(\frac{\alpha^2\sin B}{\sin A\sin C}-\frac{\beta^2\sin A}{\sin B\sin C}\right)=0, \tag{100}$$

where it is to be observed that the three points of intersection are the reciprocal points (i. e. such that $\alpha\alpha'=\beta\beta'=\gamma\gamma'$), of the intersections of parallels to the opposite sides drawn through the vertices.

We might also consider parabolæ cutting each other orthogonally at three points in the following manner. Let the curves be

$$\left.\begin{array}{l}u^2+2lu+2mv+n=0,\\ v^2+2l'u+2m'v+n'=0,\end{array}\right\} \tag{101}$$

where $u = x\cos\alpha + y\sin\alpha$, $v = x\cos\alpha - y\sin\alpha$, then expressing that these cut orthogonally we obtain a hyperbola of the form

$$Auv + Bu + Cv + D = 0. \tag{102}$$

Now let the origin be taken at the centre of this hyperbola, then

$$B = C = 0, \text{ or } m + l\cos 2\alpha = 0,\ l' + m'\cos 2\alpha = 0. \tag{103}$$

We thus get $uv = k^2$, where $k^2 = lm'\cos 2\alpha \sin^2 2\alpha$.

Hence from (101) we must express that the equations

$$u^3 + 2lu^2 + nu + 2mk^2 = 0,$$
$$2l'u^3 + n'u^2 + 2m'k^2u + k^4 = 0,$$

coincide. We may thus take $l = b$, $m = -b\cos\theta$, $l' = -a\cos\theta$, $m' = a$, $n = n' = -4ab\cos\theta$, where $2\alpha = \theta$, and $\cos\theta = \sqrt{5} - 2$; that is, the parabolæ

$$\left.\begin{aligned} u^2 + 2b(u - v\cos\theta) - 4ab\cos\theta = 0, \\ v^2 + 2a(v - u\cos\theta) - 4ab\cos\theta = 0, \end{aligned}\right\} \tag{104}$$

where $u = x\cos\alpha + y\sin\alpha$, $v = x\cos\alpha - y\sin\alpha$, $\theta = 2\alpha$, and $1 + 2\sin\alpha = \sqrt{5}$, cut orthogonally at three of their points of intersection.

If a parabola be given and we wish to find another parabola cutting it orthogonally at three points, we may proceed as follows: taking circular coordinates we write the first curve $\sqrt{x} + \sqrt{y} = 1$, or

$$x^2 + y^2 - 2xy - 2x - 2y + 1 = 0, \tag{105}$$

and the second

$$A\lambda^2 + B\mu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 0, \tag{106}$$

in tangential coordinates. The circle passing through the foci is then

$$2(G - F)xy - Ay + Bx = 0. \tag{107}$$

Expressing that this must be identical with the circle S_{12}, namely,

$$C'_{12}xy - F_{12}x - G_{12}y + H_{12} = 0, \tag{108}$$

we obtain

$$F - G = 2\lambda,\ s + G + \lambda A = 0, \tag{109}$$

$$s + F^2 - \lambda B = 0,\ s = AB - H^2 = H(F + G) - AF - BG + FG. \tag{109}$$

Putting $F = \theta + \lambda$, $G = \theta - \lambda$, we get $A + B = 4\theta$, and hence we take $A = 2\theta + \phi$, $B = 2\theta - \phi$, so that we have

$$2\theta H = 2\theta^2 + \lambda\phi, \text{ and } 16\theta^4 + 4\theta^2(\lambda^2 - \phi^2) = \lambda^2\phi^2,$$

which gives $\lambda^2 + 4\theta^2 = 0$, the factor $4\theta^2 - \phi^2$ being neglected.

Thus the parabola (106) becomes

$$(2\theta+\phi)\lambda^2+(2\theta-\phi)\mu^2+2\lambda\mu(\theta+i\phi)$$
$$+2\theta\mu\nu(1+2i)+2\theta\nu\lambda(1-2i)=0, \quad (110)$$

where $i^2=-1$. This represents a system of parabolæ having the axis in common with the parabola

$$3x^2+3y^2+10xy-2(x+y)+3+4i(y-x)(y+x-3)=0, \quad (111)$$

and touching this same parabola at the point of contact of a tangent inclined at an angle of 45° to the axis of the parabola (105). The curve (111), it may be observed, in Cartesian coordinates, is

$$(4x+2y)^2-4x-24y+3=0,$$

the parabola (105) at the same time being $4y^2=4x+1$. From the results just obtained, we see that two parabolæ cannot cut orthogonally at more than three points. If the parabola $y^2-px=0$ is cut orthogonally by a conic U at four points, U must be the ellipse $y^2+2x^2=\alpha$, in which case two of the points are real and two imaginary. It may be observed that $y^2=x$ is cut orthogonally at three points by the conic

$$y^2+2x^2+2\lambda xy+\lambda^2(2x-1)+2\lambda(1+\lambda^2)y=0. \quad (112)$$

Again, I observe that if the angles of the triangle of reference in trilinear coordinates are such that $1+\cos A+\cos B+\cos C=0$, then the equation

$$(\beta^2-\gamma^2)^2\cos B\cos C+(\gamma^2-\alpha^2)^2\cos C\cos A+(\alpha^2-\beta^2)^2\cos A\cos B=0 \quad (113)$$

represents two equilateral hyperbolæ cutting each other orthogonally at the centres of the three circles escribed to the sides. In the case of conics cutting each other orthogonally at three points, we have, referring them to the triangle formed by the points,

$$\left.\begin{aligned}U_1&=l_1\beta\gamma+m_1\gamma\alpha+n_1\alpha\beta,\\ U_2&=l_2\beta\gamma+m_2\gamma\alpha+n_2\alpha\beta\end{aligned}\right\} \quad (114)$$

and

$$l_1l_2+m_1m_2=(l_1m_2+l_2m_1)\cos C, \quad m_1m_2+n_1n_2=(m_1n_2+m_2n_1)\cos A,$$
$$n_1n_2+l_1n_2=(l_1n_2+l_2n_1)\cos B, \quad (115)$$

Eliminating l_2, m_2, n_2, we get

$$(\cos B\cos C-\cos A)\,l\,(m^2+n^2)+(\cos C\cos A-\cos B)\,m\,(n^2+l^2)$$
$$+(\cos A\cos B-\cos C)\,n\,(l^2+m^2)+2(1-\cos A\cos B\cos C)\,lmn=0. \quad (116)$$

If l, m, n are tangential coordinates, this shows that the lines

$$\frac{\alpha}{l_1}+\frac{\beta}{m_1}+\frac{\gamma}{l_1}=0,\quad \frac{\alpha}{l_2}+\frac{\beta}{m_2}+\frac{\gamma}{n_3}=0 \tag{117}$$

passing through the points where the tangents of U_1, U_2 respectively, at the vertices, meet the opposite sides, are corresponding tangents of the curve of the third class (116), namely, tangents such that the line joining the points of contact touches the curve.

Again, if l, m, n are the trilinear coordinates of a point, U_1, U_2 are the polar conics of such points with regard to the triangle and (116) shows that the points are corresponding points of a cubic curve passing through the vertices, through the points where

$$\frac{\alpha}{\cos A-\cos B\cos C}+\frac{\beta}{\cos B-\cos A\cos C}+\frac{\gamma}{\cos C-\cos A\cos B}=0 \tag{118}$$

meets the sides, and through the middle points of the perpendiculars. If one of the angles is right, (116) breaks up into factors. Taking $C=90°$, we get the factors

$$n=l\sin A+m\sin B, \tag{119}$$

$$\frac{\sin A}{l}+\frac{\sin B}{m}-\frac{1}{n}=0. \tag{120}$$

In the first case, the three points are the three vertices of a rectangle inscribed in the curve, and in the second case, the line $\frac{\alpha}{l}+\frac{\beta}{m}+\frac{\gamma}{n}=0$ is at right angles to the side γ opposite the right angle.

By combining the equation (116) with the condition $l_1\cos A+m_1\cos B+n_1\cos C=0$ that U_1 should be an equilateral hyperbola, we get the three equilateral hyperbolæ described through the vertices of the triangle so as to cut orthogonally a conic passing through these points. If the orthogonal conic is an equilateral hyperbola, that is if two equilateral hyperbolæ can cut each other orthogonally at the three points, the angles of the triangle must be such that

$$1=\sin^2 A+\sin^2 B+\sin^2 C. \tag{121}$$

Again, combining the equation (116) with

$$\sqrt{(l\sin A)}+\sqrt{(m\sin B)}+\sqrt{(n\sin C)}, \tag{122}$$

the condition that U should be a parabola, we get the six parabolæ that can be

described through the three points to cut orthogonally other conics at three points. A few observations can be made with regard to conics having a focus in common. If the conics

$$x^2 + y^2 = (ax + by + k)^2, \quad x^2 + y^2 = (a'x + b'y + k)^2 \tag{123}$$

cut orthogonally at two points on the chord of intersection

$$(a + a')x + (b + b')y + 2k = 0, \tag{124}$$

the relation

$$aa' + bb' + 1 = 0 \tag{125}$$

is satisfied. In this case the latera recta of the conics are equal and the eccentricities are such that $ee' \cos\theta = 1$, where θ is the angle between the directrices. With regard to the angles of intersection of two such conics, I observe that from geometrical considerations we see that the angle of intersection at a point P is equal to half the angle subtended at P by the two other foci H, H'. Let the distances of a point from H, H' be ρ_1, ρ_2, and let α, β be the corresponding directrices, then the conics are $\rho_1^2 = e_1^2\alpha^2$, $\rho_2^2 = e_2^2\beta^2$. Suppose α, β at right angles to each other, then the four points of intersection lie on the circle

$$\frac{\rho_1^2}{e_1^2} + \frac{\rho_2^2}{e_2^2} = \rho^2, \tag{126}$$

where ρ is the distance of a point from the intersection O of α, β. Now, suppose that this circle passes through H, H', then $e_1 = \dfrac{c}{a}$, $e_2 = \dfrac{c}{b}$, where a, b, c are the sides of the triangle O, H, H'. Subject to these three conditions, the angle $HOH' = 30°$ and the conics intersect at all their points of intersection at an angle of 30°.

With regard to the concentric conics

$$ax^2 + by^2 + 2hxy + c = 0, \; a'x^2 + b'y^2 + 2h'xy + c' = 0, \tag{127}$$

I observe that if $a'b + b'a - 2hh' = 0$, or the asymptotes are harmonically connected, then

$$\cot\theta = \frac{c(a' + b') + c'(a + b)}{\sqrt{\{c'^2(ab - h^2) + c^2(a'b' - h'^2)\}}} \tag{128}$$

where θ is the angle of intersection at four points.

Again let as consider the angle of intersection of the parabolæ

$$\rho_1^2 = (x - \alpha)^2 + (y - \beta)^2 = x^2, \; \rho_2^2 = (x - \alpha')^2 + (y - \beta')^2 = y^2, \tag{129}$$

with rectangular directrices. These curves intersect on

$$\rho_1^2 + \rho_2^2 = R^2, \tag{130}$$

which is a circle passing through the foci, if the foci and the intersection of the directrices form an equilateral triangle. From geometrical considerations we have then $2\phi = \omega + \frac{\pi}{2}$, where ϕ is the angle of intersection at a point P, and ω is the angle subtended at P by the foci. But from (130) $\omega = 30°$, which gives $\phi = 60°$, that is, if two parabolæ are such that their directrices are rectangular and the triangle formed by the foci and the intersection of the directrices is equilateral, then they cut at an angle of $60°$ at their four points of intersection.